Equilibrium layers and wall turbulence

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In turbulent flow past rigid boundaries, there can be distinguished regions close to the wall in which the local rates of energy production and dissipation are so large that aspects of the turbulent motion concerned with these processes are determined almost solely by the distribution of shear stress within the region and are independent of conditions outside it. These regions are here called equilibrium layers because of the equilibrium existing between local rates of energy production and dissipation. Three kinds of equilibrium layer have been studied experimentally, the constant-stress layer, the transpiration layer and the zero-stress layer, but there are other possible forms. One that is of importance in the theory of self-preserving flow in boundary layers and in diffusers is the 'linear-stress' layer in which the stress increases linearly with distance from the wall. The properties of these various equilibrium layers are considered and the distributions of mean velocity are derived from the equation for the turbulent kinetic energy and certain assumptions of flow similarity.

The theory of self-preserving wall flow, usually expressed as a combination of the law of the wall and the defect law, assumes compatibility between the outer flow and the equilibrium layer, and the course of development depends on the kind of equilibrium layer. Earlier work by the author, which assumed the defect law, is only valid if the whole of the equilibrium layer is a constant-stress layer and this is not true in strong adverse pressure gradients. A consistent theory is developed for these flows by assuming a 'linear-stress' layer, and the solutions show the relation between flows of finite stress and of zero stress and provide a plausible explanation of the phenomenon of downstream instability observed by Clauser. Self-preserving flow in wedges is treated on similar lines.

1. Introduction

So much has been written about the defects of the mixing-length theories of turbulent flow that it can be forgotten that the momentum-transfer form of this theory provides a simple and accurate description of the mean flow near a rigid boundary, but the usefulness of this limited success depends on establishing the conditions for its validity. The more fundamental objections to the general validity of the mixing-length approximation concern not so much the crudity of the assumed mixing process as the dependence of mixing length and eddy transport on local conditions in the flow, and they are supported by observations that the turbulent kinetic energy at a point may depend as much on transport processes from remote parts of the flow as on local production and dissipation. These objections that a turbulent flow is an integrated whole and not an assembly of quasi-independent flows do not apply to important aspects of the flow near a rigid boundary where the balance of turbulent kinetic energy is virtually unaffected by the nature of the flow in adjacent regions. Layers in which these conditions are satisfied to an acceptable approximation will be called equilibrium layers and it will be assumed that they possess an essential universality of structure which shows itself as a simple dependence of mean velocity gradient on Reynolds stress and distance from the boundary, i.e. apparent validity of the momentum-transfer theory. The best-known example is the constant-stress layer whose structure has been studied in great detail by Laufer (1955), Klebanoff (1956) and many others; other forms are the 'zero-stress' layer (Stratford 1959*a*, *b*) and the equilibrium layers on porous walls with transpiration (e.g. Dorrance & Dore 1954; Black & Sarnecki 1958), but this does not exhaust the possibilities. The purpose of this paper is to set out the necessary conditions for the existence of an equilibrium layer and to derive a slightly more general relation between velocity gradient and shear stress than the mixing-length relation. With this, a unified description of the layers is possible and this description is applied to correct and extend the theory of the development of self-preserving boundary layers in adverse pressure gradients (Townsend 1956*a*, *b*). A similar theory may be developed for self-preserving flow in conical or wedge-shaped diffusers.

2. Notation

Two-dimensional mean flows are described in terms of rectangular co-ordinates with the direction of mean flow in the xOy plane and with the Ox axis either parallel to the wall (in a boundary layer) or in the plane of symmetry (in wedge flow). Axisymmetric mean flow is described in polar co-ordinates with Ox the axis of symmetry. Then

$U, V, 0$ are the components of the mean velocity,
u, v, w are the components of the velocity fluctuation,
$q^2 = u^2+v^2+w^2$,
p is the fluctuation from the mean pressure P,
ν is the kinematic viscosity,
ϵ is the local rate of destruction of turbulent energy by viscous forces,
τ is the local Reynolds stress,
K is the Kármán constant,
B is a diffusion constant, and
K_0 is a constant characteristic of zero-stress layers.

Kinematic stresses and pressures are used, i.e. the mechanical quantities divided by the fluid density. Values at the rigid boundary are distinguished by the suffix 0 and in the free stream or on the axis of symmetry by the suffix 1.

3. Energy equilibrium in wall turbulence

In the momentum-transfer form of the mixing-length theory, velocity gradient and Reynolds stress are related by

$$\tau = -\overline{uv} = l^2 \frac{\partial U}{\partial y}\left|\frac{\partial U}{\partial y}\right|, \tag{3.1}$$

where l is a length whose magnitude must be inferred by dimensional or other considerations, and it is fairly well known that an equivalent relation may be derived from the equation for the kinetic energy of the velocity fluctuations assuming local energy and structural equilibrium (Rotta 1953). The equation is, to the usual boundary-layer approximation,

$$U\frac{\partial(\frac{1}{2}\overline{q^2})}{\partial x} + V\frac{\partial(\frac{1}{2}\overline{q^2})}{\partial y} + \overline{uv}\frac{\partial U}{\partial y} + \frac{\partial}{\partial y}(\overline{\tfrac{1}{2}q^2 v} + \overline{pv}) + \epsilon' = 0 \tag{3.2}$$

and $\epsilon' = -\nu[\overline{u\nabla^2 u} + \overline{v\nabla^2 v} + \overline{w\nabla^2 w}]$ is almost identical with the local rate of conversion of turbulent energy to heat, ϵ, in the fully turbulent part of the flow. The first two terms represent the net effect of advection of turbulent energy by the mean flow, and the *first* requirement for an equilibrium layer is that they should be negligible compared with the rate of generation of turbulent energy, $-\overline{uv}(\partial U/\partial y)$. The fourth term is the net rate of energy loss by turbulent diffusive movements and by working against turbulent pressure gradients. If this term is also negligible (the magnitude of this term and its effect on the energy balance is discussed below), the energy equation becomes

$$-\overline{uv}\frac{\partial U}{\partial y} = \epsilon, \tag{3.3}$$

indicating local equilibrium between the production and dissipation of turbulent energy.

Dissipation of energy in turbulent flow depends on working against viscous forces caused by intense velocity gradients of very small scale, and these gradients are maintained by a continuous process of stretching vortex-lines by diffusive movements. Much theoretical and experimental work shows that the rate is independent of the viscosity of the fluid and is determined by the components of the motion that contribute most to the turbulent energy and to the Reynolds stress (Batchelor 1953). Another characteristic of turbulent flow is that prolonged unidirectional shear leads to the attainment of a condition of structural similarity in which velocity fluctuations at different points in the flow are statistically similar.* If this is true, the local motion is specified by a scale of velocity and a scale of length, and it follows from dimensional considerations that

$$\epsilon = (\overline{q^2})^{\frac{3}{2}} L_\epsilon^{-1} \tag{3.4}$$

and that

$$|-\overline{uv}| = a_1 \overline{q^2}, \tag{3.5}$$

* The experimental evidence for structural similarity is derived mostly from measurements of free turbulence or of pipe and channel turbulence outside the equilibrium layers (Townsend 1956*a*, pp. 77, 152, 179, 204, 212, 253), and direct evidence for its existence in equilibrium layers is very incomplete. Some of the difficulties are mentioned in §8.

using $(\overline{q^2})^{\frac{1}{2}}$ as the scale of velocity and L_ϵ as the scale of length. Combining equations (3.3) to (3.5), we get

$$\frac{\partial U}{\partial y} = (a_1^{\frac{3}{2}} L_\epsilon)^{-1} (-\overline{uv})^{\frac{1}{2}}, \tag{3.6}$$

which is equivalent to the mixing-length result if $l = a_1^{\frac{3}{2}} L_\epsilon$.

The relation (3.6) between gradient of mean velocity and Reynolds stress becomes useful if the dissipation length L_ϵ can be inferred as a function of position in the flow, and it is usual to assume that

$$l = a_1^{\frac{3}{2}} L_\epsilon = Ky, \tag{3.7}$$

where $K \approx 0{\cdot}41$ is the Kármán constant. Dimensional reasoning confirms this assumption if (i) the scale of the motion is unaffected by the width of the whole flow, and if (ii) the scale of motion is unaffected by length scales characteristic of the stress distribution in the equilibrium layer. The first of these conditions may be satisfied by requiring that the equilibrium layer occupies only a small fraction of the total width of the flow (a *second* condition for an equilibrium layer), but the second can only be a working hypothesis justified by results. In physical terms, this condition could be satisfied if the Reynolds stress at any point were caused mostly by contributions from eddies whose scales are comparable with distance of the point from the wall since they all extend to the wall and are, in a sense, attached to it. Further, the distribution of Reynolds stress must be such that it can be produced by a possible size distribution of these eddies, and the only layer for which this is obviously possible is one of constant stress.

So far, the effects of lateral transport of turbulent energy, represented by the term $\partial(\frac{1}{2}\overline{q^2v} + \overline{pv})/\partial y$, have been neglected, but the hypothesis of structural equilibrium indicates that

$$\tfrac{1}{2}\overline{q^2v} + \overline{pv} = -a_2(\overline{q^2})^{\frac{3}{2}} \operatorname{sgn}(\partial\overline{q^2}/\partial y), \tag{3.8}$$

where a_2 is a constant of order one, and the sign of $\partial\overline{q^2}/\partial y$ has been introduced to ensure that the net energy flux is down the gradient of turbulent intensity. Substituting in the energy equation and omitting the advection terms, we obtain

$$\frac{\partial U}{\partial y} = \frac{\tau^{\frac{1}{2}}}{Ky}\left(1 - B\frac{y}{\tau}\left|\frac{\partial \tau}{\partial y}\right|\right), \tag{3.9}$$

where $B = \frac{3}{2}Ka_2a_1^{-\frac{3}{2}}$. This equation should be valid in the fully turbulent part of an equilibrium layer, i.e. in a region defined by the conditions

$$\left.\begin{array}{lll}
\text{(i)} & & \left|U\dfrac{\partial(\frac{1}{2}\overline{q^2})}{\partial x} + V\dfrac{\partial(\frac{1}{2}\overline{q^2})}{\partial y}\right| \ll -\overline{uv}\dfrac{\partial U}{\partial y}, \\
\text{or} & & \tfrac{1}{2}a_1\left|U\dfrac{\partial \tau}{\partial x} + V\dfrac{\partial \tau}{\partial y}\right| \ll \tau\dfrac{\partial U}{\partial y} \\
\text{and} & \text{(ii)} & y \ll D, \text{ where } D \text{ is the total width of the flow.}
\end{array}\right\} \tag{3.10}$$

If these conditions are satisfied, no restrictions need be imposed on the magnitudes of the mean flow acceleration and of the longitudinal pressure gradient.*

* If $B(y/\tau)|\partial\tau/\partial y| > 1$, equation (3.9) is meaningless and no equilibrium layer is possible. This is most unlikely to occur in unidirectional flow.

4. Distributions of mean velocity in equilibrium layers

Within the region determined by the conditions (3.10), mean velocity distributions are obtained by substitution of the stress distribution in equation (3.9). Near an impermeable boundary, the stress distribution may be approximated by

$$\tau = \tau_0 + \alpha y, \tag{4.1}$$

and then integration of the equation (3.9) with this stress distribution leads to

$$U = \frac{\tau_0^{\frac{1}{2}}}{K}\left[\log\frac{(\tau_0+\alpha y)^{\frac{1}{2}} - \tau_0^{\frac{1}{2}}}{(\tau_0+\alpha y)^{\frac{1}{2}} + \tau_0^{\frac{1}{2}}}\right] + \frac{2(1-B\operatorname{sgn}\alpha)}{K}(\tau_0+\alpha y)^{\frac{1}{2}} + U_t. \tag{4.2}$$

The constant of integration, U_t, may be regarded as a slip velocity or velocity of translation that has no effect on the motion in the fully turbulent flow and its magnitude may be expected to depend on the nature of the surface and on τ_0, α and ν. If the surface is smooth and at rest and if the layer is fully turbulent at distances from the wall small compared with τ_0/α, the velocity distribution (4.2) must be identical with the ordinary logarithmic distribution for constant stress

$$U = \frac{\tau_0^{\frac{1}{2}}}{K}\left[\log\frac{\tau_0^{\frac{1}{2}} y}{\nu} + A\right] \tag{4.3}$$

for small values of $\alpha y/\tau_0$. For these small values, (4.2) may be written

$$U = \frac{\tau_0^{\frac{1}{2}}}{K}\left[\log\frac{\alpha y}{4\tau_0} + 2(1-B\operatorname{sgn}\alpha)\right] + U_t,$$

and so

$$U_t = \frac{\tau_0^{\frac{1}{2}}}{K}\left[\log\frac{4\tau_0^{\frac{3}{2}}}{\alpha\nu} + A - 2(1-B\operatorname{sgn}\alpha)\right]$$

and

$$U = \frac{\tau_0^{\frac{1}{2}}}{K}\left[\log\left(\frac{4\tau_0^{\frac{3}{2}}}{\alpha\nu}\,\frac{(\tau_0+\alpha y)^{\frac{1}{2}} - \tau_0^{\frac{1}{2}}}{(\tau_0+\alpha y)^{\frac{1}{2}} + \tau_0^{\frac{1}{2}}}\right) + A - 2(1-B\operatorname{sgn}\alpha)\right] + \frac{2(1-B\operatorname{sgn}\alpha)}{K}(\tau_0+\alpha y)^{\frac{1}{2}}. \tag{4.4}$$

The condition that the layer is fully turbulent at a distance from the wall small compared with τ_0/α is met if $\alpha y \ll \tau_0$ at the inner edge of the turbulent flow, i.e. at $y = 20\nu/\tau_0^{\frac{1}{2}}$, the requirement being that $\tau_0^{\frac{3}{2}}(\alpha\nu)^{-1} \gg 20$.

The best-known equilibrium layer of this class, the constant-stress layer, has been studied in great detail, especially by Laufer (1955)* and by Klebanoff (1956), and there is little doubt that in it the basic requirements of energy equilibrium and universality of structure are nearly satisfied so far as the motions responsible for momentum and heat transfer are concerned. In most flows along solid walls, a constant-stress layer may be distinguished in which the mean velocity varies in the way given by equation (4.3) but, if the flow is opposed by a strong adverse pressure gradient, the constant-stress region forms only a small part of the total equilibrium layer defined by the conditions (3.10). Even though the

* In a pipe or channel, the stress variation is linear, but the thickness of the equilibrium layer is small compared with the channel width, and, to the same approximation, the stress variation is negligible.

stress gradient depends on the varying flow acceleration as well as the constant (with y) longitudinal pressure gradient, (4.1) is often a satisfactory approximation to the stress distribution and the distribution of mean velocity is given by equation (4.4). This equation takes a more convenient form if $\alpha y \gg \tau_0$ (α necessarily positive):

$$U = \frac{2(1-B)}{K}(\alpha y)^{\frac{1}{2}} + \frac{\tau_0^{\frac{1}{2}}}{K}\left[\log\frac{4\tau_0^{\frac{3}{2}}}{\alpha\nu} + A - 2(1-B)\right], \tag{4.5}$$

which differs from the velocity distribution in a layer of zero wall stress,

$$U = \frac{2}{K_0}(\alpha y)^{\frac{1}{2}} + C(\alpha\nu)^{\frac{1}{3}} \tag{4.6}$$

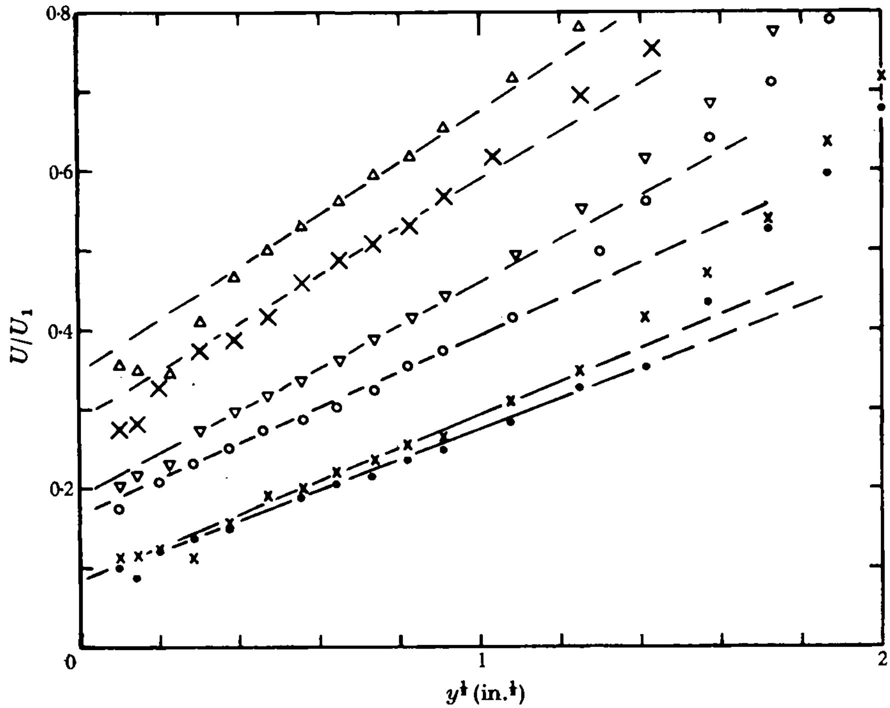

FIGURE 1. Distributions of mean velocity in a turbulent boundary layer near separation (from Schubauer & Klebanoff 1951). ●, $x = 25{\cdot}4$ ft.; ×, $x = 25{\cdot}0$ ft.; ○, $x = 24{\cdot}5$ ft.; ▽, $x = 24{\cdot}0$ ft.; ╳, $x = 23{\cdot}0$ ft.; △, $x = 22{\cdot}0$ ft. N.B. Separation occurs between $x = 25{\cdot}5$ ft. and $x = 26{\cdot}0$ ft. Reynolds number $= 10^6$ ft.$^{-1}$ Dashed curves indicate limits to validity of approximations.

(Stratford 1959*a*; Ellison 1960), only in the additive constant which represents a 'slip' velocity at the edge of the layer. Because of the condition $\tau_0^{\frac{3}{2}}(\alpha\nu)^{-1} \gg 20$, equation (4.5) does not hold for very small values of τ_0, and its limiting form as τ_0 approaches zero is slightly different from (4.6). Notice that if the viscosity is very small, i.e. the Reynolds number is large, $\log(4\tau_0^{\frac{3}{2}}/\alpha\nu)$ becomes large and a comparatively small wall stress may produce a large slip velocity.

Experimental evidence of the occurrence of velocity distributions of the form (4.5) can be found in the measurements by Schubauer & Klebanoff (1951) of flow in a turbulent boundary layer immediately upstream of the position of separation of flow from the surface. In figure 1, some of their measurements of mean velocity are plotted against the square root of distance from the surface, and extensive linear regions are found for the positions, $x = 24{\cdot}5$, $25{\cdot}0$, $25{\cdot}4$ ft., and less extensive ones at positions further upstream. The region of validity of the logarithmic distribution (4.3) does not extend beyond $y = 0{\cdot}08$ in. although the total thickness of the layer is between 6 and 7 in. These measurements may be used to estimate the 'zero-stress' constant K_0, but uncertainty in the magnitudes of the flow accelerations makes the calculation difficult.* Using smoothed values of the acceleration at the wall-end of the linear regions and the measured slopes, it is found that

$$K_0 = 0{\cdot}48 \pm 0{\cdot}03,$$

which may be compared with the value of 0·50 (Townsend 1960) based on measurements by Stratford (1959*b*) in a self-preserving flow of zero wall stress. Since $K_0 = K/(1-B)$, the diffusion constant B is about 0·2 and is of the expected sign.

Another kind of equilibrium layer is found on a porous wall though which fluid is moving with an average velocity V_0. If the longitudinal pressure gradient is not too large, the acceleration of the mean flow will also be small and the momentum equation reduces to

$$V_0 \frac{\partial U}{\partial y} = \frac{\partial \tau}{\partial y},$$

which integrates to

$$\tau - UV_0 = \tau_0, \tag{4.7}$$

showing that the total flux of momentum from the wall is independent of y. Substituting in the basic equation (3.9) and integrating, we obtain

$$\frac{2K}{V_0}[(\tau_0 + UV_0)^{\frac{1}{2}} - \tau_0^{\frac{1}{2}}] + B \operatorname{sgn} V_0 \log\left(1 + \frac{UV_0}{\tau_0}\right) = \log y + \text{const.}, \tag{4.8}$$

where the constant of integration is expected to depend on the nature of the surface and on V_0, τ_0 and ν. For small values of UV_0/τ_0, the distribution must become identical with the logarithmic distribution, and so

$$\frac{2K}{V_0}[(\tau_0 + UV_0)^{\frac{1}{2}} - \tau_0^{\frac{1}{2}}] + B \operatorname{sgn} V_0 \log\left(1 + \frac{UV_0}{\tau_0}\right) = \log \frac{\tau_0^{\frac{1}{2}} y}{\nu} + A. \tag{4.9}$$

Except for the term $B \log\{1 + (UV_0/\tau_0)\} \operatorname{sgn} V_0$, which is usually small, this equation is that derived from the mixing-length theory by Dorrance & Dore (1954) and by others, and it has been confirmed by measurements of velocity distributions in boundary layers with suction and blowing (Black & Sarnecki 1958).

5. Equilibrium layers and self-preserving flow

A turbulent flow is self-preserving in structure if the motions at different sections of the flow differ only in scales of velocity and length, and are dynamically similar in those aspects of the motion that control the distributions of mean

* The flow accelerations are negative and equal to about one-third of the longitudinal pressure gradient.

velocity and Reynolds stress. The importance of self-preserving flows in the theory of turbulence is that the governing equations become ordinary differential equations and that predictions of growth and of friction are then possible, but only a few types of flow can be exactly self-preserving. A greater number can develop in a nearly self-preserving way and it is useful to distinguish two ways in which this is possible. An example of the first kind is the wake of a cylinder in a uniform stream in which the distribution of mean velocity is of the form

$$\left.\begin{aligned} U &= U_1 - u_0 f(y/l_0) \\ \text{and the Reynolds stress} \quad \overline{uv} &= u_0^2 g(y/l_0), \end{aligned}\right\} \tag{5.1}$$

where l_0 and u_0 are scales of length and velocity depending only on x, and the functions $f(\eta)$ and $g(\eta)$ are characteristic of the whole flow. Substitution in the equation of mean motion leads to an ordinary differential equation with solutions satisfying the condition of constant momentum flux only if $l_0 \propto (x-x_0)^{\frac{1}{2}}$ and $u_0 \propto (x-x_0)^{-\frac{1}{2}}$.* In this kind of self-preserving flow, eddy structures and mean velocity distributions develop in a natural and unforced way from similar forms further upstream, and the advection terms in the momentum and energy equations are of the same order of magnitude as those representing local effects such as stress gradient or rate of energy production. The other kind of self-preserving development is one of nearly absolute equilibrium and would occur in flow in a channel whose width changed very slowly. Then the flow at each section would adjust itself to the local width and, conforming to the principle of Reynolds number similarity would have velocity and stress distributions of the self-preserving form (5.1). The condition for this kind of flow is that advection terms should be negligible in the energy equation.

Considered as a whole, but independently of its outer flow, an equilibrium layer with the linear stress distribution (4.1) is self-preserving with length scale τ_0/α and velocity scale $\tau_0^{\frac{1}{2}}$, and the whole of the flow may be self-preserving if these same scales are suitable for self-preserving development of the outer flow. This requirement is very restrictive in boundary-layer flow unless either τ_0 or α is negligibly small. If these conditions can be satisfied, exactly or approximately, the course of development may be obtained from the equation of mean motion in terms of the velocity distribution function. This function has the form given by the similarity arguments in the equilibrium layer and a good approximation to it in the outer flow may be obtained by assuming

(i) that the Reynolds stress is related to velocity gradient by a coefficient of eddy viscosity depending on x alone, i.e.

$$-\overline{uv} = \nu_T \frac{\partial U}{\partial y}, \tag{5.2}$$

(ii) that its magnitude is determined by the condition

$$\frac{1}{\nu_T}\int (U_1 - U)\, dy = R_s, \tag{5.3}$$

where R_s is a constant characteristic of the kind of flow (boundary layer, channel, pipe, etc.),

* Exactly true only if $U_1 - U \ll U_1$.

(iii) that the velocity distribution in the outer layer joins smoothly the velocity distribution in the inner layer. Clauser (1956) and Townsend (1956*a*,*b*, 1960) describe some experimental measurements that confirm these assumptions.

6. Boundary layers in adverse pressure gradients

To the usual boundary-layer approximation, the equation of mean motion for a boundary layer is

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+\frac{\partial \overline{uv}}{\partial y}=U_1\frac{dU_1}{dx}, \tag{6.1}$$

where U_1 is the velocity in the free stream and two-dimensional mean flow is assumed. Substitution of the self-preserving forms for the distribution of mean velocity and Reynolds stress (5.1) gives

$$-\frac{d(u_0U_1)}{dx}f+\frac{u_0}{l_0}\frac{d(U_1l_0)}{dx}\eta f'+u_0\frac{du_0}{dx}f^2-\frac{u_0}{l_0}\frac{d(u_0l_0)}{dx}f'\int_0 f\,d\eta+\frac{u_0^2}{l_0}g'=0, \tag{6.2}$$

which is an ordinary differential equation with independent variable $\eta = y/l_0$ if the coefficients are in constant ratio. It follows that $dl_0/dx=$ constant, $u_0/U_1=$ constant, $(l_0/u_0)(du_0/dx)=$ constant, but the scales so defined must be compatible with the scales of the equilibrium layer, $u_0=\tau_0^{\frac{1}{2}}$, $l_0=\tau_0/\alpha$, which is only possible either

(*a*) if $U_1 \propto (x_0-x)^{-1}\propto \tau_0^{\frac{1}{2}}$, $l_0\propto(x_0-x)$ (Townsend 1956*a*),

or (*b*) if $\tau_0=0$, $U_1\propto(x-x_0)^{-0\cdot 23}$, $l_0\propto(x-x_0)$ (Stratford 1959*a*,*b*; Townsend 1960).

The first is an accelerating flow in a converging wedge and the second a zero-stress flow in an adverse pressure gradient.

The converging flow may be regarded as the prototype of a family of approximately self-preserving layers which share two of its characteristics, an equilibrium layer of nearly constant stress and an outer layer with small velocity defects $[(U_1-U)\ll U_1]$ at high Reynolds numbers. If the distribution of velocity in the equilibrium layer is given by the logarithmic equation (4.3), the velocity defect always becomes small at sufficiently large Reynolds numbers and self-preserving flow in the pressure distribution defined by $U_1\propto(x-x_0)^a$ is possible if $a>-\frac{1}{3}(x>x_0)$. For values of a close to $-\frac{1}{3}$ the wall stress is very small and the pressure gradient large so that a large variation of stress in the equilibrium layer is inevitable (Dunn 1956).

Flows in these strong adverse pressure gradients must resemble more closely the zero-stress self-preserving flow, and we now consider nearly self-preserving flows with equilibrium layers whose stress distribution is nearly linear with a characteristic length τ_0/α small compared with the thickness and a velocity distribution given by equation (4.5). The distribution of free stream velocity is assumed to be $U_1\propto(x-x_0)^a$ and, as in the zero-stress layer, the scales of velocity and length are the free stream velocity U_1 and $(x-x_0)$. Substituting these scales in equation (6.2) yields

$$2af-(a+1)\eta f'-af^2+(a+1)f'\int_0 f\,d\eta-g'=0, \tag{6.3}$$

which has a self-preserving solution if the inner boundary condition can be expressed in a form independent of x. Within the equilibrium layer and not too close to the wall, the velocity distribution function is

$$f(\eta) = \zeta + \frac{2}{K_0}\left[\frac{\alpha(x-x_0)}{U_1^2}\eta\right]^{\frac{1}{2}}, \tag{6.4}$$

where

$$\zeta = \frac{\tau_0^{\frac{1}{2}}}{KU_1}\left[\log\frac{4\tau_0^{\frac{3}{2}}}{\alpha\nu} + A - 2(1-B)\right], \tag{6.5}$$

and the flow is self-preserving if ζ and $\alpha(x-x_0)/U_1^2$ are constant. It will be seen that this can not be exactly true but the variation of ζ is very slow and it is reasonable to suppose that the flow adjusts itself to the local value of ζ and resembles the hypothetical self-preserving flow of constant ζ over moderate ranges of x.

The wall stress is related to the velocity distribution and the pressure gradient by the equation for the momentum integral,

$$\frac{d}{dx}\int_0^\infty U(U_1-U)\,dy + \frac{dU_1}{dx}\int_0^\infty (U_1-U)\,dy = \tau_0, \tag{6.6}$$

or in terms of the function $f(\eta)$,

$$(2a+1)(I_a - I_b) + aI_a = \tau_0/U_1^2, \tag{6.7}$$

where

$$I_a = \int_0^\infty f(\eta)\,d\eta, \quad I_b = \int_0^\infty [f(\eta)]^2\,d\eta.$$

This relation is consistent with the assumption of self-preserving flow if the variation of τ_0/U_1^2 is small compared with $-aI_a$.

The velocity distribution function is now obtained using the eddy-viscosity assumptions of the last section. Making use of equations (5.2) and (5.3), the distribution function in the outer layer satisfies

$$2af - (a+1)\eta f' - af^2 + (a+1)f'\int_0 f\,d\eta + I_a f''/R_s = 0, \tag{6.8}$$

a form of the Falkner–Skan equation. For flows in strong adverse pressure gradients, the wall stress is small compared with the pressure force on the layer, i.e. $\tau_0/U_1^2 \ll -aI_a$, and may be neglected except in so far as it determines ζ. An approximate solution of (6.8) satisfying the boundary conditions

$$f(0) = C, \quad f'(0) = 0, \quad f''(0) = R_s Ca(2-C)/I_a$$

is

$$f(\eta) = C\exp\left(-\tfrac{1}{2}R^2\eta^2\right), \tag{6.9}$$

where

$$I_a R^2 = -aR_s(2-C). \tag{6.10}$$

For a smooth junction with the velocity distribution in the equilibrium layer at $\eta = \eta_1$,

$$1 - C\exp\left(-\tfrac{1}{2}R^2\eta_1^2\right) = \zeta + 2K_0^{-1}(\alpha_0\eta_1)^{\frac{1}{2}},$$

$$CR^2\eta_1\exp\left(-\tfrac{1}{2}R^2\eta_1^2\right) = K_0^{-1}(\alpha_0/\eta_1)^{\frac{1}{2}}, \tag{6.11}$$

where $\alpha_0 = -a(1-\zeta^2)$ very nearly (see Appendix). Approximately, if $R^2\eta_1^2$ is not too large,

$$I_a = \left(\tfrac{1}{2}\pi\right)^{\frac{1}{2}} CR^{-1} + 2\alpha_0/3K_0^2R^2,$$

$$I_b = \tfrac{1}{2}\pi^{\frac{1}{2}}C^2R^{-1} + \frac{4\alpha_0}{3K_0^2R^2} + \frac{2\alpha_0}{3K_0^2}\frac{\alpha_0^{\frac{3}{2}}}{(K_0CR^2)^{\frac{3}{2}}}. \tag{6.12}$$

The solutions of these equations are conveniently presented by plotting the friction coefficient τ_0/U_1^2 as a function of ζ for various values of the exponent a (from equations (6.7), (6.10), (6.11) and (6.12)) and as a function of ζ for various values of $-3aR_x$ (substituting values of ζ obtained from equations (6.10), (6.11) and (6.12) in equation (6.5)). This has been done to slide-rule accuracy in figure 2, in which possible self-preserving flows are represented by intersections of the two families of curves, e.g. in the pressure gradient defined by $a = -0{\cdot}24$, the friction coefficient is nearly $\tau_0/U_1^2 = 9{\cdot}6\times10^{-4}$ at a Reynolds number of 10^6. The two bounding curves indicate the limits of validity of the approximations used, the left-hand curve imposing the condition that the constant-stress part of the equilibrium layer is fully turbulent and the right-hand one that the wall friction plays a small part in the momentum balance.*

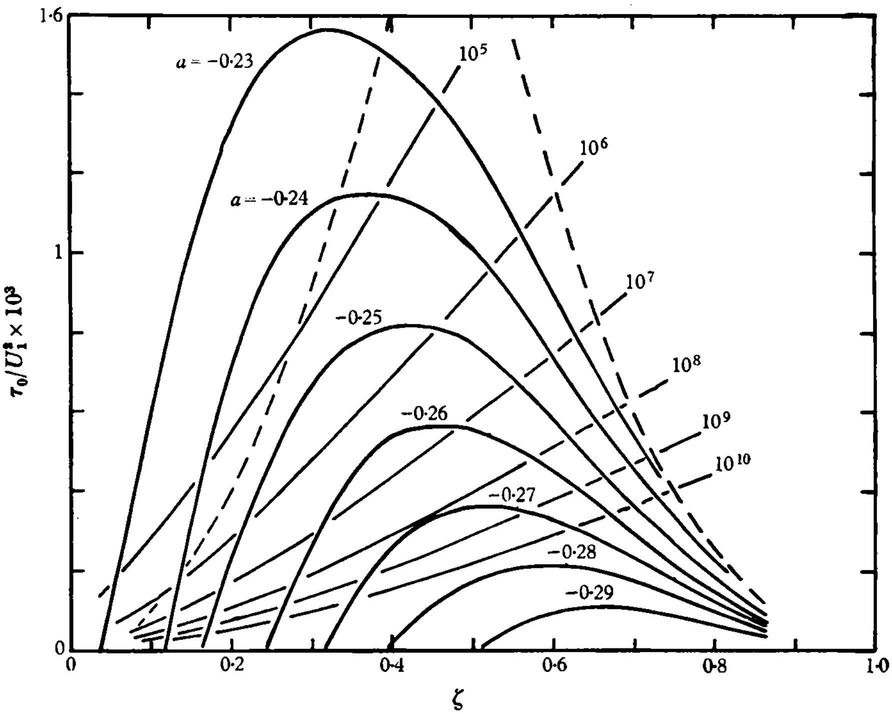

FIGURE 2. Boundary-layer development with a 'linear-stress' equilibrium layer. The numbers on the 'upper' family of curves refer to values of the exponent a, and on the lower family to values of $-3aR_x$ which is not much different from R_x.

The most interesting feature of these solutions is that two different layers are possible, one with comparatively large wall stress and small velocity defect and another with low stress and large defect, but this is only consistent with the known existence of a zero-stress layer for $a \approx -0{\cdot}23$ (Stratford 1959*b*) and of finite-stress layers for this and more negative exponents (Clauser 1956). In figure 3, the friction coefficient is shown as a function of Reynolds number for

* Conditions for internal consistency of the 'linear-stress' approximation are derived in the Appendix. It turns out that they are satisfied if $\tau_0/U_1^2 \ll -3aI_a$.

flows of constant exponent and it is seen that, although flow at infinite Reynolds number is possible with exponents as negative as $-\frac{1}{3}$, the limiting value at ordinary Reynolds numbers is very much less, changing from $-0{\cdot}25$ near $R_x = 3\times 10^5$ to $-0{\cdot}28$ near $R_x = 10^{11}$. It is interesting to compare this variation of the friction coefficient with the dependence in a flow constrained by a continuous, slight adjustment of the pressure gradient to obey the defect law

$$U = U_1 - \tau_0^{\frac{1}{2}} F(y/\delta) \tag{6.13}$$

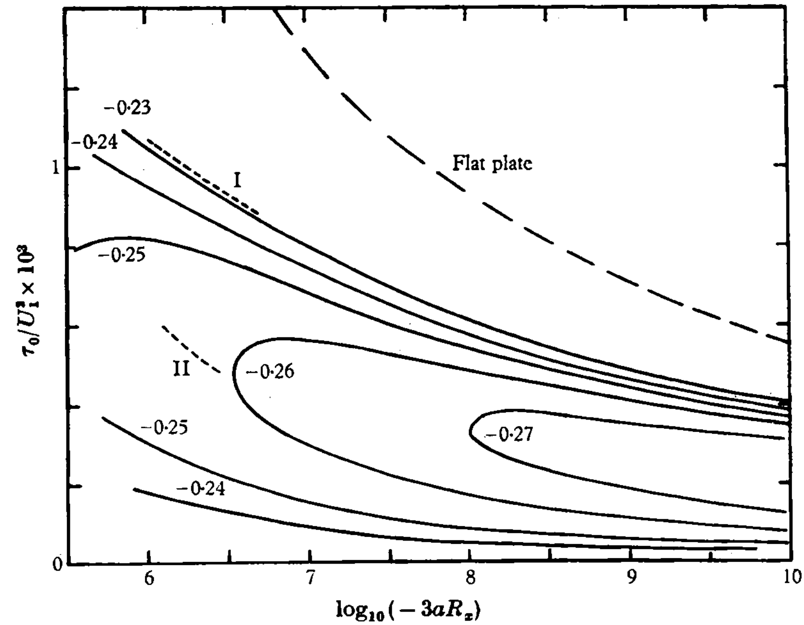

FIGURE 3. Variation of wall stress with Reynolds number for a boundary layer with a constant exponent. (The short broken lines represent measurements by Clauser in pressure distributions I and II.)

in the outer flow (figure 4). If development takes place along the upper branch of the friction curve, the necessary adjustment is small and development with constant exponent might be indistinguishable experimentally from development with constant velocity-defect ratio, but along the lower branch this is not true. On this branch, the defect constant C approaches a constant value as the Reynolds number increases and the wall stress tends to zero comparatively rapidly.

Some comparisons of these theoretical predictions with the measurements of Clauser (1954, 1956) are given in table 1. If it is remembered that uncertainty in the position of the flow origin leads to considerable uncertainty in the appropriate values of R_x and a, the agreement is quite satisfactory. Lines showing the observed variation of friction with Reynolds number have been drawn on figure 3 and it is clear that flow I is developing along the upper branch. Flow II is in a region of ambiguous development and its observed development is not described by this theory. However, Clauser adjusted the pressure gradient so that the outer flow obeyed the defect law (6.13) which, in this region, requires an 'unnatural' course of development, and it is not surprising that the maintenance of this development was difficult and that, unlike most self-preserving flows, small deviations tended to grow rather than to disappear. Some remarks of Stratford (1959*b*) suggest that flow on the lower branch is stable in this sense and so this

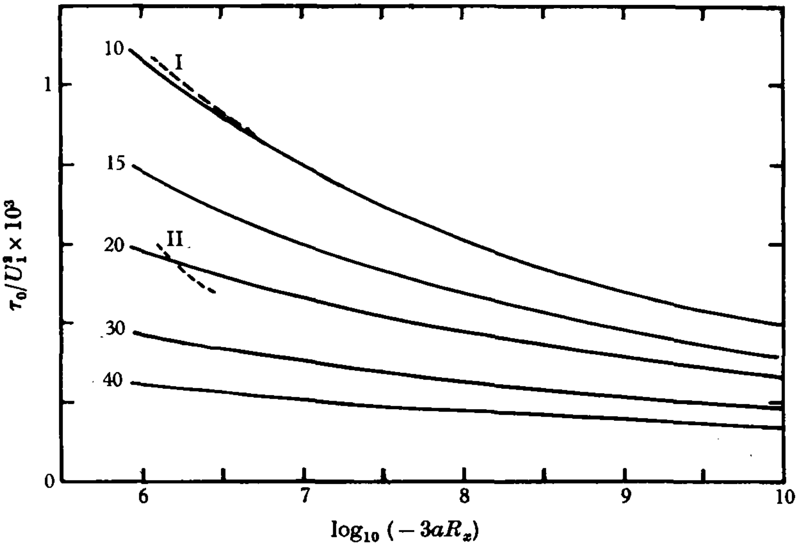

FIGURE 4. Variation of wall stress with Reynolds number for constant velocity-defect ratio, $U_1 C/\tau_0^{\frac{1}{2}}$, in the outer layer. The numbers near the curves are the corresponding values of the ratio. (N.B. The measured defect ratio for distribution I is 12, for distribution II, 24.)

	Observed				Theory		
Flow	τ_0/U_1^2	$\frac{\delta^*}{\tau_0}\frac{dP_1}{dx}$	$-3aI_a$	C	τ_0/U_1^2	a	C
I	$9{\cdot}5\times10^{-4}$	2	$5{\cdot}7\times10^{-3}$	0·39	$9{\cdot}6\times10^{-4}$	−0·24	0·374
					8·5	−0·245	0·374
II	5·0	7	10·5	0·53	6·0	−0·256	0·491
					5·3	−0·258	0·491

N.B. The theoretical values refer to two values of Reynolds number, 0·75 and $1{\cdot}5\times10^6$, and the observed values of $-3aI_a$.

TABLE 1

explanation of this phenomenon of downstream instability has the virtue that self-preserving development remains a stable asymptotic state for turbulent flow.

The calculations that are presented in the diagrams have assumed that $K = 0{\cdot}41$, $K_0 = 0{\cdot}50$, $B = 0{\cdot}2$, $R_s = 75$, values which have been found to describe flow in a zero-stress layer (Townsend 1960). The value of R_s also agrees closely with the value necessary to describe boundary-layer development in zero pressure gradient by the procedure of § 5.

7. Self-preserving flow between diverging planes

The Reynolds number of two-dimensional mean flow between converging or diverging plane boundaries, based on local velocity and channel width, has the same value everywhere and so it is possible that the motion is dynamically similar at all sections of the flow and that the flow is self-preserving. We consider flow in a wedge of semi-angle θ, with the Oz axis the apex of the wedge and the xOz plane the central plane of symmetry. To the boundary-layer approximation, applicable if the angle θ is small, the equation of mean motion may be written

$$-f^2+g' = -\mathscr{P}+\nu M^{-1}f'', \tag{7.1}$$

self-preserving flow being implied by the use of the distribution functions, f and g and the pressure recovery coefficient, $\mathscr{P}$, defined by

$$\left.\begin{aligned} U = Mf(\eta)/x, \quad V = M\eta f(\eta)/x, \quad \eta = y/x,\\ \overline{uv} = M^2x^{-2}g(\eta), \quad \mathscr{P} = \frac{x^3}{M^2}\frac{dP_1}{dx}. \end{aligned}\right\} \tag{7.2}$$

It is convenient to define M so that $f(0) = 1$, i.e. $M/x = U_1$, the mean velocity in the central plane at distance x from the apex. The Reynolds number of the flow is

$$R_x = U_1 x/\nu = M/\nu.$$

Integrating equation (7.1) from the centre line to the wall,

$$\int_0^\theta f^2 d\eta = \mathscr{P}\theta + \frac{\tau_0}{U_1^2}, \tag{7.3}$$

where τ_0 is the shear stress on the walls of the diffuser.

For sufficiently large Reynolds numbers, the converging flow takes the form of two boundary layers separated by a central region of negligible Reynolds stress,* but a diverging flow must be turbulent over the whole width of the channel. We now apply to the diverging flow the same procedure that has been used in § 6 to obtain the mean-flow properties of self-preserving boundary layers. Outside the equilibrium layers, the mean velocity distribution is found by assuming a constant coefficient of eddy viscosity, given by

$$R_s = \frac{1}{\nu_T}\int_0^{\theta x}(U_1-U)\,dy = \frac{U_1 x}{\nu_T}I_a, \tag{7.4}$$

where R_s is a constant characteristic of fully developed flow between plane boundaries. Measurements by Laufer (1951) of pressure flow between parallel planes indicate that $R_s = 28 \pm 3$. Within the central flow then, we have

$$f^2 + (I_a/R_s)f'' = \mathscr{P}, \tag{7.5}$$

and the solution of this equation for the boundary conditions $f(0) = 1$, $f'(0) = 0$ is

$$\eta = \left[\frac{2I_a}{R_s(1-\mathscr{P})}\right]^{\frac{1}{2}}\beta_1\int_0^{X/\beta_1}\frac{dt}{[(1-t^2)(1-k^2t^2)]^{\frac{1}{2}}}, \tag{7.6}$$

* In a rapidly accelerating flow, the only strong sources of vorticity are the walls and so boundary layers develop.

where β_1^2, β_2^2 are the roots of $x^2-3x+3(1-\mathscr{P})=0$, and

$$k=\beta_1/\beta_2 \quad (\beta_1<\beta_2) \quad \text{and} \quad X^2=1-f(\eta).$$

The integral in (7.6) is tabulated as the elliptic integral of the first kind,

$$F(x)=\int_0^x \frac{dt}{[(1-t^2)(1-k^2t^2)]^{\frac{1}{2}}}$$

and its inverse function, sn, is defined by

$$\operatorname{sn}[F(x)]=x.$$

The velocity distribution function is

$$f(\eta)=1-\beta_1^2\operatorname{sn}^2\left[\left\{\frac{R_s(1-\mathscr{P})}{2I_a}\right\}^{\frac{1}{2}}\frac{\eta}{\beta_1}\right], \tag{7.7}$$

and a good approximation to the flow integral I_a is

$$\begin{aligned} I_a &= \int_0^\theta \{1-f(\eta)\}\,d\eta = \left[\frac{2I_a}{R_s(1-\mathscr{P})}\right]^{\frac{1}{2}}\beta_1^3\int_0^{u_1}\operatorname{sn}^2 u\,du \\ &= \left[\frac{2I_a}{R_s(1-\mathscr{P})}\right]^{\frac{1}{2}}\beta_1^3 k^{-2}(F-E), \end{aligned} \tag{7.8}$$

where F, E are the elliptic integrals of the first and second kinds evaluated at $X(\theta)/\beta_1$ and

$$u_1=\left[\frac{R_s(1-\mathscr{P})}{2I_a}\right]^{\frac{1}{2}}\frac{\theta}{\beta_1}.$$

Using the relations between β_1, $\mathscr{P}$ and k implied by the definition of β_1, i.e.

$$\beta_1^2=\frac{3k^2}{1+k^2}, \quad 1-\mathscr{P}=\frac{3k^2}{(1+k^2)^2},$$

it follows that

$$R_sI_a=\frac{18}{1+k^2}(F-E)^2 \tag{7.9}$$

and the wall stress is given by

$$\frac{\tau_0}{U_1^2}=-\frac{I_a}{R_s}f'(\theta)=\frac{18}{R_s}\frac{k^2}{(1+k^2)^2}(F-E)\operatorname{sn}u_1\operatorname{cn}u_1\operatorname{dn}u_1. \tag{7.10}$$

Both these relations depend on the approximation that the difference between the true value of the flow integral, I_a, and the value found by assuming the central velocity distribution (7.7) to extend all the way to the wall is negligible. These equations relate $\mathscr{P}$ and τ_0 if θ and R_s are known, and their values are found by imposing conditions for a smooth junction of the central velocity distribution with the velocity distribution in the equilibrium layer.

For a linear variation of stress in the equilibrium layer, $\tau=\tau_0+\alpha y$ where $\alpha y \gg \tau_0$ for most of the layer, and

$$f(\eta)=\zeta+2K_0^{-1}[(\mathscr{P}-\zeta^2)(\theta-\eta)]^{\frac{1}{2}}, \tag{7.11}$$

since the stress gradient in the layer is nearly $(\mathscr{P}-\zeta^2)U_1^2x^{-1}$ (see Appendix). The conditions for a smooth junction are that

$$\left.\begin{aligned} \zeta+2K_0^{-1}[(\mathscr{P}-\zeta^2)(\theta-\eta_1)]^{\frac{1}{2}} &= 1-\beta_1^2\operatorname{sn}^2u_1, \\ \text{and that}\quad \frac{1}{K_0}\left(\frac{\mathscr{P}-\zeta^2}{\theta-\eta_1}\right)^{\frac{1}{2}} &= \frac{R_s}{I_a}(\mathscr{P}-\zeta^2)(\theta-\eta_1)+\frac{R_s}{I_a}\frac{\tau_0}{U_1^2}. \end{aligned}\right\} \tag{7.12}$$

The assumption of linear stress implies that $\operatorname{sn} u_1 = 1$ and that

$$(\mathscr{P}-\zeta^2)(\theta-\eta_1) \gg \tau_0/U_1^2,$$

so, eliminating $\theta-\eta_1$, we have

$$\zeta+\frac{2}{K_0}\left[\frac{I_a}{R_s K_0}(\mathscr{P}-\zeta^2)\right]^{\frac{1}{2}} = 1-\beta_1^2, \tag{7.13}$$

essentially a relation between ζ and $\mathscr{P}$. Now ζ is related to the wall stress by

$$\zeta = \frac{\tau_0^{\frac{1}{2}}}{KU_1}\left[\log\left(\frac{4\tau_0^{\frac{3}{2}}}{U_1^3}\frac{R_x}{\mathscr{P}-\zeta^2}\right)+A-2(1-B)\right], \tag{7.14}$$

and the condition $\operatorname{sn} u_1 = 1$ can be expressed as

$$R_s\theta = 6K(K-E), \tag{7.15}$$

so that the wall stress can be calculated as a function of flow Reynolds number and angle of divergence. The results of these calculations are shown in figure 5 with an indication of the limit of validity of the approximation that $\tau_0 \ll \alpha y_1$. Validity of the approximation that the stress distribution is substantially linear can be expressed by the condition

$$10\theta^{\frac{2}{3}}\zeta \ll 1$$

(see (A. 20) in the Appendix). $10\theta^{\frac{2}{3}}\zeta$ is about 0·30 for $\theta = 0{\cdot}05$, and so the results of this treatment are only applicable for larger values of θ.

A special case of some interest is the flow with zero wall stress and with $\zeta = 0$, which occurs at a critical angle of divergence θ_0. This angle depends on the values of K_0 and R_s, but the pressure recovery and $R_s\theta_0$ are functions of $K_0 R_s^{\frac{1}{2}}$ and a few values are tabulated in table 2.

Assuming $K_0 = 0{\cdot}50$, $R_s = 28$, we find $\theta_0 = 0{\cdot}075$ rad. $= 4{\cdot}3$ deg., $\mathscr{P} = 0{\cdot}53$. Milliat (1957) has reported steady diffuser flow for a semi-angle of three degrees which is not much less than this calculated value. It should be emphasized that secondary flow is very likely to occur in these flows and will cause separation of flow from the walls at smaller angles of divergence than the critical value for truly two-dimensional flow. An interesting point about the critical flow is that the pressure-recovery coefficient is less than for flows in wedges of smaller angle.

If the stress in the equilibrium layer is nearly constant,

$$f(\eta) = \frac{\tau_0^{\frac{1}{2}}}{KU_1}\left[\log\left\{\frac{\tau_0^{\frac{1}{2}}}{U_1}R_x(\theta-\eta)\right\}+A\right] \tag{7.16}$$

and the conditions for a smooth junction are

$$\left.\begin{aligned} \frac{\tau_0^{\frac{1}{2}}}{KU_1}\left[\log\left\{\frac{\tau_0^{\frac{1}{2}}}{U_1}R_x(\theta-\eta_1)\right\}+A-1\right] &= 1-\beta_1^2\operatorname{sn}^2 u_1 \\ \text{and}\qquad \frac{\tau_0^{\frac{1}{2}}}{KU_1}\frac{1}{\theta-\eta_1} &= \frac{R_s}{I_a}\frac{\tau_0}{U_1^2}, \end{aligned}\right\} \tag{7.17}$$

or, eliminating $\theta-\eta_1$,

$$\frac{\tau_0^{\frac{1}{2}}}{KU_1}\left[\log R_x+\log\frac{I_a}{KR_s}+A-1\right] = 1-\beta_1^2\operatorname{sn}^2 u_1. \tag{7.18}$$

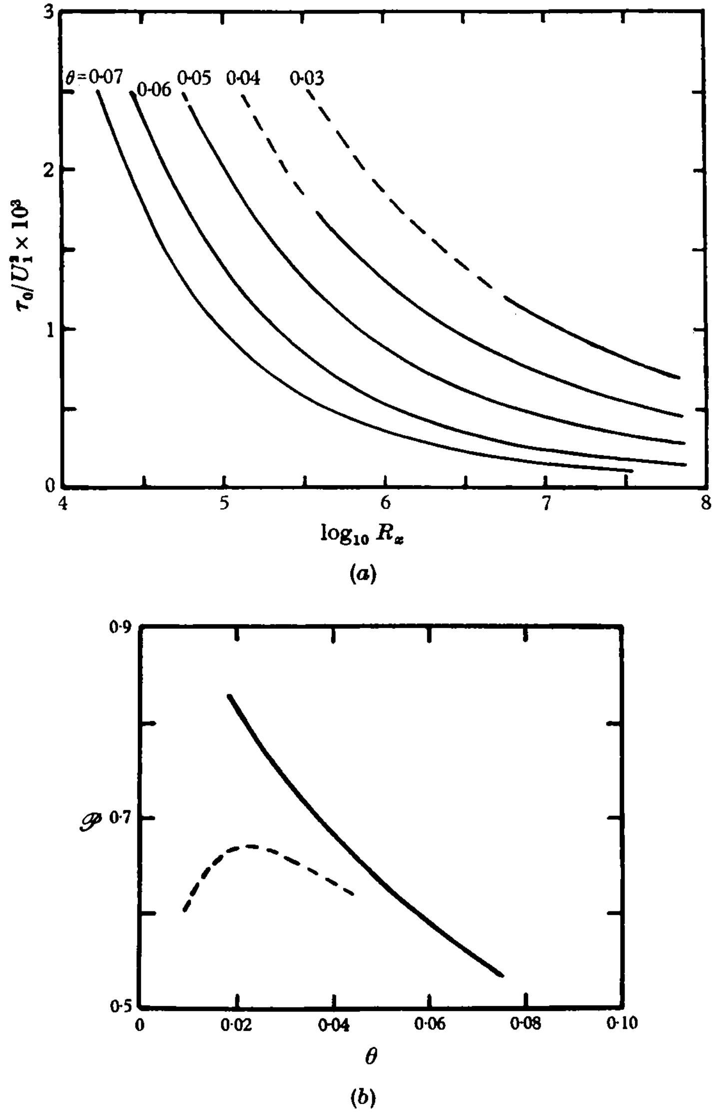

FIGURE 5. Self-preserving flow in a wedge with a 'linear-stress' equilibrium layer. (*a*) Variation of wall stress with Reynolds number for various angles of divergence. (The broken portions of the curves are outside the range of validity of the linear-stress approximation.) (*b*) Variation of pressure-recovery coefficient with angle of divergence. (The dashed curve shows, for comparison, the variation for $\theta R_x = 3 \times 10^4$ if the equilibrium layer is of nearly constant stress.)

$K_0 R_s^{\frac{1}{2}}$	$\mathscr{P}$	$\theta_0 R_s$
1·64	0·718	0·935
2·12	0·614	1·492
2·83	0·520	2·21
4·26	0·441	3·11

TABLE 2

This equation and equations (7.9) and (7.10) determine τ_0/U_1^2 and $\mathscr{P}$ as functions of R_x for a given θ, and the results of solving these equations are shown graphically in figure 6. The approximation of constant stress is valid if the corresponding velocity distribution could be maintained with a small variation of stress in the equilibrium layer and the condition for this, derived in the Appendix, is

$$\tau_0/U_1^2 > 0{\cdot}2\theta^{\frac{4}{3}}.$$

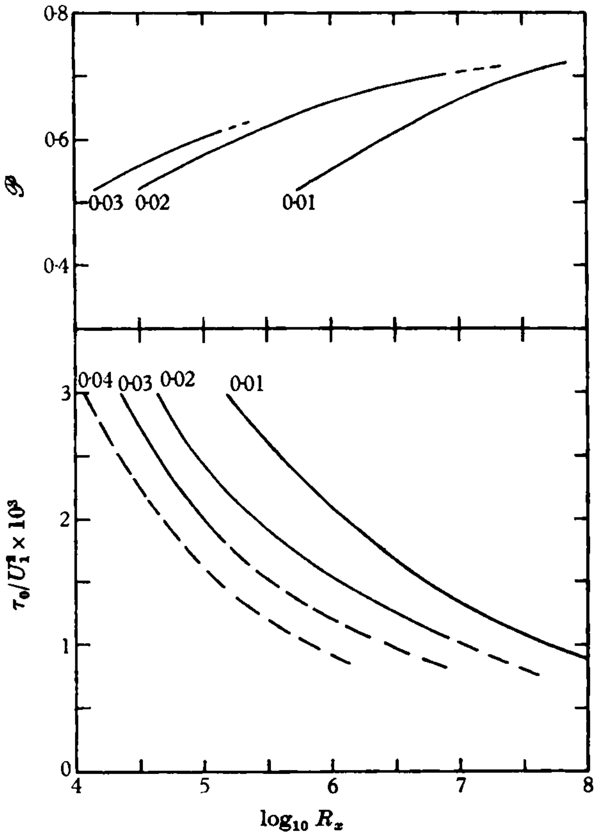

FIGURE 6. Self-preserving flow in a wedge with a 'constant-stress' equilibrium layer. (The broken portions of the curves are outside the range of validity of the constant-stress approximation.)

It follows that the constant-stress approximation will fail at very high Reynolds numbers however small the angle of divergence, and that the increase in pressure-recovery coefficient with Reynolds number has a limit.

Ruetenik (1954, see also Ruetenik & Corrsin 1955) and Craya & Milliat (1955) have made studies of the self-preserving turbulent flow in a wedge of semi-angle one degree, each at a Reynolds number of nearly $1{\cdot}8 \times 10^6$. Reference to figure 6 shows that these conditions are within the range of validity of the constant-stress approximation, and that the theory predicts a friction coefficient $\tau_0/U_1^2 = 15 \times 10^{-4}$, and a pressure coefficient $\mathscr{P} = 0{\cdot}67$. The two sets of measurements are in good general agreement but Ruetenik gives rather more detail. He finds that

$\tau_0/U_1^2 = 13 \pm 1{\cdot}3 \times 10^{-4}$, and $\mathscr{P} = 0{\cdot}57$ and his measured distributions of mean velocity and of Reynolds stress are compared with the theoretical distributions in figure 7. Excepting the large difference between the observed and predicted values of $\mathscr{P}$, general agreement is found and the discrepancies can be attributed to uncertainty in the value of the flow constant R_s. A considerable part of the error

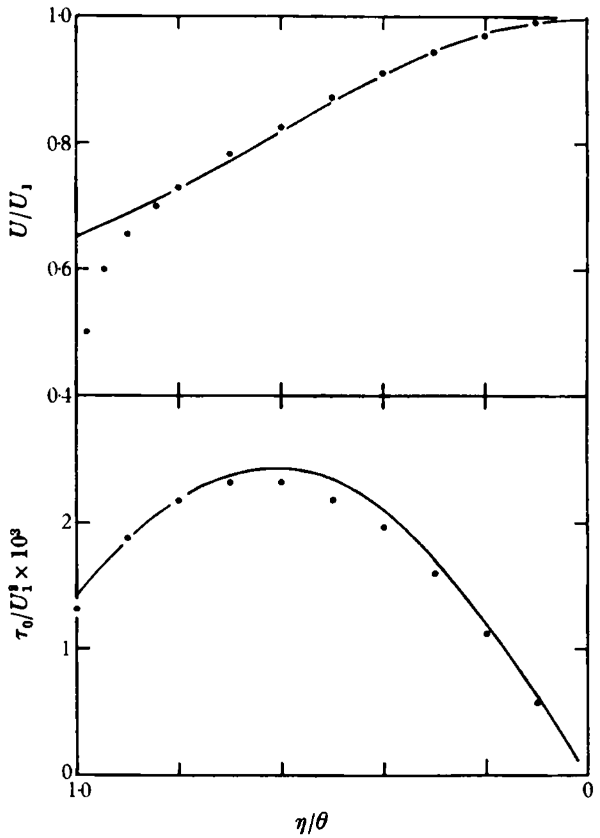

FIGURE 7. Comparison of theoretical predictions with measurements by Ruetenik (1954) of velocity and stress distributions in a diverging wedge of semi-angle one degree. (Experimental results ●, theory ——.)

in $\mathscr{P}$ is due to assuming that integrals over the whole flow can be approximated by replacing the composite velocity distribution by the velocity distribution for the central flow extrapolated to the walls. This procedure underestimates I_a and overestimates the integral $\int_0^\theta f^2(\eta)\,d\eta$ by about 10%. Since

$$\mathscr{P}\theta = \int_0^\theta f^2 d\eta - \frac{\tau_0}{U_1^2},$$

the overestimate is reflected in the value of $\mathscr{P}$. The necessary modification of the equations is straightforward but their solution becomes even more tiresome.

8. Discussion

The essential property of an equilibrium layer is local and absolute equilibrium between energy supply and dissipation, and definition by this property rather than by stress equilibrium extends considerably the applicability of similarity arguments. If we are interested only in the mean-velocity distribution inside the equilibrium layer, the only gain over simple acceptance of the momentum-transfer theory is a plausible explanation of the difference between the Kármán constant and the corresponding constant for zero-stress flow, but the requirement of energy equilibrium sets clear limits to the extent of the layer which are not provided by the mixing-length theory. It is interesting that the energy equilibrium is of the same kind as that assumed in the theory of local similarity, the difference being that one is localized in physical space and the other in wave-number space. Unlike the motion described by the theory of local similarity, the motion in an equilibrium layer has an influence on all parts of the flow, and the second half of this paper is an attempt to develop a method for solving the mean-flow problem in wall flows with different kinds of equilibrium layers. The origin of the method is an observation of Clauser (1956) that the velocity distribution in the outer part of self-preserving boundary layers is nearly that produced by an eddy viscosity of constant ratio to the integrated velocity defect at the particular section. Although the final results are not very sensitive to this assumption, comparison with experiment suggests that the eddy-viscosity hypothesis gives a good description of the mean-velocity distribution and that the constant ratio may depend only on the conformation of the bounding surfaces and not on the pressure gradient.

The hypothesis that the *turbulent* motion in an equilibrium layer is in a universal state determined by the stress distribution is not confirmed by numerous observations that turbulent intensities in constant-stress layers vary considerably between different flows of the same stress. This is in strong contrast with the universality of the distributions of mean velocity and it is difficult to reconcile these observations without supposing that the motion at any point consists of two components, an active component responsible for turbulent transfer and determined by the stress distribution and an inactive component which does not transfer momentum or interact with the universal component. This does not mean that the eddy structures contributing to the inactive component at a particular point can not form part of the active component at points further from the wall, and it seems likely that the inactive motion is a meandering or swirling motion made up from attached eddies of large size which contribute to the Reynolds stress much further from the wall than the point of observation. If this is true, the ratio of turbulent intensity to Reynolds stress would increase with total thickness of the constant-stress layer, and comparison of the ratios in the very thick boundary layer on the earth's surface and in laboratory boundary layers confirms this.

REFERENCES

Batchelor, G. K. 1953 *Homogeneous Turbulence.* Cambridge University Press.
Black, T. J. & Sarnecki, A. J. 1958 *Aero. Res. Counc., Lond.*, Rep. no. 20,501.
Clauser, F. H. 1954 *J. Aero. Sci.* **21**, 91.
Clauser, F. H. 1956 *Advanc. App. Mech.* **4**, 1.
Craya, A. & Milliat, J.-P. 1955 *C.R. Acad. Sci., Paris*, **241**, 542.
Dorrance, W. H. & Dore, F. J. 1954 *J. Aero. Sci.* **21**, 404.
Dunn, J. W. 1956 *Math. Rev.* **17**, 1249.
Ellison, T. H. 1960 *J. Fluid Mech.* **8**, 33.
Jeffreys, H. & Jeffreys, B. S. 1950 *Methods of Mathematical Physics.* Cambridge University Press.
Klebanoff, P. S. 1956 *Nat. Adv. Comm. Aero., Wash.*, Rep. no. 1247.
Laufer, J. 1951 *Nat. Adv. Comm. Aero., Wash.*, Rep. no. 1033.
Laufer, J. 1955 *Nat. Adv. Comm. Aero., Wash.*, Rep. no. 1174.
Milliat, J.-P. 1957 *9me. Congrès intern. Mécan. appl., Univ. Bruxelles*, **3**, 419.
Rotta, J. 1953 *Ing.-Arch.* **19**, 31.
Ruetenik, J. R. 1954 *Johns Hopk. Univ., Inst. Co-op. Res.*, Rep. no. 1–19.
Ruetenik, J. R. & Corrsin, S. 1955 50 *Jahre Grenzschichtforschung*, p. 446. (Ed. H. Görtler and W. Tollmien.) Vieweg: Braunschweig.
Schubauer, G. B. & Klebanoff, P. S. 1951 *Nat. Adv. Comm. Aero., Wash.*, Rep. no. 1030.
Stratford, B. S. 1959*a* *J. Fluid Mech.* **5**, 1.
Stratford, B. S. 1959*b* *J. Fluid Mech.* **5**, 17.
Townsend, A. A. 1956*a* *Structure of Turbulent Shear Flow.* Cambridge University Press.
Townsend, A. A. 1956*b* *J. Fluid Mech.* **1**, 561.
Townsend, A. A. 1960 *J. Fluid Mech.* **8**, 143.

Appendix: stress distributions in equilibrium layers

The procedure described in §5 gives good results only if the simplified distribution of stress in the equilibrium layer, assumed for mathematical convenience, bears some resemblance to the real distribution. Close resemblance requires that the stress distribution computed from the assumed velocity distribution and the inferred rate of development of the layer should be nearly the same as the assumed stress distribution. If the stress variation in the equilibrium layer is assumed to be small, the velocity distribution is given by the logarithmic profile, and it is easy to show that the distribution of stress must be

$$\tau-\tau_0 = \frac{dP_1}{dx}y+\frac{1}{2K^2}\frac{d\tau_0}{dx}y\left[\left(\log\frac{\tau_0^{\frac{1}{2}}y}{\nu}+A-1\right)^2+1\right] \tag{A.1}$$

within the equilibrium layer. The original assumption of constant stress is only valid if $\tau-\tau_0$ so computed is small compared with τ_0.

If a self-preserving boundary layer has an equilibrium layer of nearly constant stress, the velocity defect outside the equilibrium layer is small at sufficiently large Reynolds numbers, and wall stress and Reynolds number are related by

$$R_x = (1+3a)K^{-3}I_1\gamma^{-2}\exp(\gamma^{-1}-A), \tag{A.2}$$

where $\gamma = \tau_0^{\frac{1}{2}}/(KU_1), \quad I_1 = \int_0^\infty K\frac{U_1-U}{\tau_0^{\frac{1}{2}}}d\eta,$

and $$\eta = \frac{\tau_0^{\frac{1}{2}} y}{\nu} \exp(A - \gamma^{-1}).$$

Assuming a constant eddy viscosity in the outer layer, the velocity distribution function is

$$f(\eta) = K(U_1 - U)/\tau_0^{\frac{1}{2}} = CHh_n(R\eta), \tag{A.3}$$

where $$R = \frac{KR_s^{\frac{1}{2}}}{I_1}\left(\frac{1+a}{1+3a}\right)^{\frac{1}{2}}, \quad n = \frac{2a}{1+a}$$

and the function Hh_n is defined by

$$Hh_n(x) = \int_0^\infty \frac{t^n}{n!} \exp[-\tfrac{1}{2}(t+x)^2]\,dt, \quad \frac{d}{dx}Hh_n(x) = -Hh_{n-1}(x) \tag{A.4}$$

(Jeffreys & Jeffreys 1950, p. 622). In the constant-stress equilibrium layer, $f(\eta) = -\log\eta$ and the integral I_1 is nearly

$$I_1 = CR^{-1}Hh_{n+1}(0) + \eta_1, \tag{A.5}$$

where η_1 specifies the junction of the two layers. The conditions for a smooth junction are approximately

$$1 - CHh_n(0) = \log\eta_1, \quad CRHh_{n-1}(0) = \eta_1^{-1}, \tag{A.6}$$

so that $$I_1 CR = KR_s^{\frac{1}{2}}\left(\frac{1+a}{1+3a}\right)^{\frac{1}{2}} C = \frac{\pi^{\frac{1}{2}} C^2}{2^{\frac{1}{2}n+1}(\frac{1}{2}n+\frac{1}{2})!} + \pi^{-\frac{1}{2}} 2^{\frac{1}{2}n}(\tfrac{1}{2}n - \tfrac{1}{2})!. \tag{A.7}$$

The three equations (A. 5), (A. 6) and (A. 7) determine C, R and I_1 as functions of the exponent a. From the friction equation (A. 2), we have

$$\frac{d\tau_0}{dx} = K^2\gamma^2 \frac{dU_1^2}{dx}[1 + O(\gamma)] \tag{A.8}$$

and, to the approximation that $\gamma \ll 1$,

$$\frac{\tau - \tau_0}{\tau_0} = \frac{2a}{3a+1} \frac{\eta_1(\log\eta_1 - 1)}{I_1}. \tag{A.9}$$

Using the values of η_1 and I_1 given by equations (A. 6) and (A. 7), we find

$$\frac{\tau - \tau_0}{\tau_0} = \frac{-2a}{[2(1+a)(1+3a)]^{\frac{1}{2}}} \frac{1}{KR_s^{\frac{1}{2}}} \frac{(\frac{1}{2}n - \frac{1}{2})!}{(\frac{1}{2}n)!}, \tag{A.10}$$

omitting the small second term on the right of (A. 7). This shows that the stress variation in the equilibrium layer must become very large as a approaches the critical value of $-\frac{1}{3}$, whatever the Reynolds number. For $a = -0{\cdot}2$, the ratio is nearly 0·4, and so the assumption of constant stress in the equilibrium layer must fail for more negative values of a.

A similar calculation of the stress variation in the equilibrium layer of self-preserving wedge flow leads to

$$\frac{\tau - \tau_0}{\tau_0} = \frac{I_a}{KR_s} \frac{U_1^3}{\tau_0^{\frac{3}{2}}}\left[\mathscr{P} - (1 - \beta_1^2 \operatorname{sn}^2 u_1)^2 - \frac{\tau_0}{K^2 U_1^2}\right], \tag{A.11}$$

and substitution of values of I_a, τ_0, $\mathscr{P}$, β_1 and u_1 would show whether the constant-stress approximation is permissible in any particular flow. The condition for this may be expressed more compactly and with sufficient accuracy by observing that

$$\frac{I_a}{KR_s} = \frac{\theta^2}{2K}\frac{1}{(1+k^2)F^2}$$

(equations (7.9) and (7.10)) and that $\mathscr{P}-(1-\beta_1^2\,\mathrm{sn}^2\,u_1)^2 = 0{\cdot}25$, $1+k^2 = 1{\cdot}25$, $F^2 = 1{\cdot}5$ very roughly for all the points calculated in figure 4. Substituting these approximate values and allowing a 50 % variation of stress in the equilibrium layer, the condition for an equilibrium layer of nearly constant stress is that

$$\tau_0/U_1^2 > 0{\cdot}2\theta^{\frac{4}{3}}. \tag{A. 12}$$

The contrasting approximation is that the stress variation is linear, i.e. $\tau = \tau_0 + \alpha y$ with $\alpha y \gg \tau_0$ for most of the equilibrium layer, when the velocity distribution is (equation (4.5))

$$U = U_l + 2K_0^{-1}(\alpha y)^{\frac{1}{2}}, \tag{A. 13}$$

where

$$U_l = \frac{\tau_0^{\frac{1}{2}}}{K}\left[\log\frac{4\tau_0^{\frac{3}{2}}}{\alpha\nu} + A - 2(1-B)\right].$$

The stress distribution necessary to maintain this velocity distribution is

$$\tau - \tau_0 = \left(\frac{dP_1}{dx} + U_l\frac{dU_l}{dx}\right)y + \frac{2}{3K_0\alpha^{\frac{1}{2}}}\frac{d(\alpha U_l)}{dx}y^{\frac{3}{2}} + \frac{2}{3K_0^2}\frac{d\alpha}{dx}y^2, \tag{A. 14}$$

and the linear stress distribution is a good approximation to this distribution if

$$\alpha = \frac{dP_1}{dx} + U_l\frac{dU_l}{dx} \tag{A. 15}$$

and if the last two terms on the right-hand side are small compared with the first. In a boundary layer, $dP_1/dx = -U_1\,dU_1/dx$, and $\zeta = U_l/U_1$ varies much more slowly than U_1, so the conditions for smallness become

$$\frac{2}{3K_0}\frac{1-3a}{(-a)^{\frac{1}{2}}}\frac{\zeta}{(1-\zeta^2)^{\frac{1}{2}}}\left(\frac{y_1}{x}\right)^{\frac{1}{2}} \ll 1 \quad \text{and} \quad \frac{2}{3K_0^2}(1-2a)\frac{y_1}{x} \ll 1. \tag{A. 16}$$

Substituting in the first condition the value of $\eta_1 = y_1/x$ implied by equations (6.10), (6.11) and (6.12), we obtain the condition for development with an equilibrium layer of nearly linear stress distribution

$$\frac{2(1-3a)}{-3a}\left(\frac{\pi}{2}\right)^{\frac{1}{4}}(K_0R_s^2)^{-\frac{3}{4}}\frac{\zeta}{1+\zeta} \ll 1. \tag{A. 17}$$

This condition is satisfied if the exponent defining the pressure gradient is more negative than the value for zero-stress development, $\alpha = -0{\cdot}23$. The second condition of (A. 16) imposes no additional restriction. A last requirement is that $\alpha y_1 \gg \tau_0$, necessary for validity of the velocity distribution (A. 13). Using the approximate forms of equations (6.10), (6.11) and (6.12),

$$CR = -a(2/\pi)^{\frac{1}{2}}R_s(2-C), \quad \zeta = 1-C,$$

we find that the requirement is satisfied if

$$\left(\frac{\pi}{2K_0 R_s^2}\right)^{\frac{2}{3}} (1-\zeta)^2 \gg \tau_0/U_1^2. \tag{A.18}$$

In self-preserving wedge flow, the conditions for smallness of the last two terms in (A.14) become

$$\frac{8}{3K_0}\frac{\zeta}{(\mathscr{P}-\zeta^2)^{\frac{1}{2}}}(\theta-\eta_1)^{\frac{1}{3}} \ll 1, \quad \frac{2}{K_0^2}(\theta-\eta) \ll 1. \tag{A.19}$$

Substituting the value of $(\alpha-\eta_1)$ given by the conditions for a smooth junction (7.12), we obtain

$$\frac{8}{3K_0}\frac{\zeta}{(\mathscr{P}-\zeta^2)^{\frac{1}{2}}}\left(\frac{I_a}{R_s K_0}\right)^{\frac{1}{3}} \ll 1,$$

and, using the typical values of the slowly varying quantities $\mathscr{P}-\zeta^2$, $(1+k^2)$ and F previously listed, we find that the condition is

$$10\theta^{\frac{2}{3}}\zeta \ll 1. \tag{A.20}$$

The condition that $\alpha y_1 \gg \tau_0$ may be obtained in a similar way as

$$\left[\frac{\theta^2}{2K_0 F^2(1+k^2)}(\mathscr{P}-\zeta^2)\right]^{\frac{2}{3}} \gg \tau_0/U_1^2,$$

or, using the typical values,

$$0{\cdot}26\theta^{\frac{4}{3}} \gg \tau_0/U_1^2. \tag{A.21}$$